Thm. (Contd.) To show that the mapping

$\phi : B(N) \to B(N^*)$

s.t. $\phi(T) = T^* \quad \forall\, T \in B(N) \quad \to (6)$

is an isometric isomorphism i.e. ϕ is 1-1 L.T. s.t. $\|\phi(T)\| = \|T\|$

Now $\|\phi(T)\| = \|T\| = \|T^*\|$, from (6)

Now we shall show that ϕ is linear and 1-1.

ϕ is linear — Let $T, U \in B(N)$ and α, β any scalars.

Then

$$\phi(\alpha T + \beta U) = (\alpha T + \beta U)^*, \text{ from (6)}$$

Now $[(\alpha T + \beta U)^*(f)](x) = f((\alpha T + \beta U)(x))$, from (1)

$$= f((\alpha T)(x)) + f((\beta U)(x)), \ \because f \text{ is linear}$$

$$= \alpha f(T(x)) + \beta f(U(x)), \ \because f \text{ linear}$$

$$= \alpha [T^*(f)](x) + \beta [U^*(f)](x)$$

, from (1)

$$\therefore (\alpha T + \beta U)^*(f) = \alpha [T^*(f)] + \beta [U^*(f)]$$

$$= (\alpha T^* + \beta U^*)(f)$$

$$\therefore (\alpha T + \beta U)^*(f) = \alpha T^* + \beta U^* \quad \to (7)$$

$\therefore\ \phi(\alpha T + \beta U) = (\alpha T + \beta U)^*$

$= \alpha T^* + \beta U^*$

$= \alpha \phi(T) + \beta \phi(U)$.

$\because \phi(T) = T^*$

$\therefore\ \phi$ is linear.

ϕ is 1-1 $\rightarrow$

$\phi(T) = \phi(U) \Rightarrow T^* = U^*$

$\Rightarrow \|T^* - U^*\| = 0$

$\Rightarrow \|(T-U)^*\| = 0$

; from (7), if we take $\alpha = 1$, $\beta = -1$

$\Rightarrow \|T - U\| = 0$; $\because \|T^*\| = \|T\|$ from (5)

$\Rightarrow T = U$

$\Rightarrow \phi$ is 1-1

Thus we have shown that ϕ is an isometric isomorphism.

Now we need to show that 'ϕ' preserves products and preserves the identity transformation we have—

$[(TU)^*(f)](x) = f((TU)(x))$; from (1)

$= f(T(U(x)))$

$= [T^*(f)](U(x))$, from (1) as $U(x) \in N$

$$[(TU)^*(f)](x) = [(U^*T^*)(f)](x)$$

Hence $(TU)^* = U^*T^*$ $\rightarrow (8)$

$\Rightarrow \phi(TU) = (TU)^* = U^*T^*$

$\Rightarrow \phi$ reserves product.

Now to show 'ϕ' preserves the identity transformation i.e. $\phi(I) = I^* = I$.

Let I be the identity operator on N.

Then,

$$[I^*(f)](x) = f(I(x)) \quad ;\text{from } (1)$$

$$= f(x), \quad \because I(x) = x$$

$$= (If)(x), \quad \text{as } If = f$$

$\therefore [I^*(f)](x) = (If)(x)$

$\Rightarrow I^*(f) = (If)$

$\Rightarrow I^* = I$

so that $\phi(I) = I^* = I$

$\therefore$ 'ϕ' preserves the identity transformation

Hence the theorem.